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The Characteristic Morphism of an Algebra

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Abstract This paper is motivated by the observation that the characteristic morphism of an algebra relates to certain smoothness condition closely. We show that for an algebra A of finite global dimension, if the characteristic morphism is injective, then A has finite Hochschild cohomology dimension. In particular, if A is semi-simple, then the characteristic morphism is injective if and only if A is homologically smooth. Moreover, the characteristic morphism of a finite dimensional path algebra is injective. Recall that a path algebra is always homologically smooth.

Keywords Hochschild cohomology · Graded center · Characteristic morphism

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1 Introduction

Let $\Bbbk$ be a field and A an associative algebra over $\Bbbk$. There exists an algebra homomorphism, called the characteristic morphism, from the Hochschild cohomolgy of A to its derived center, the graded center of the derived category. In this paper, we mainly discuss the injectivity of characteristic morphisms, and show how it relates to the homological smoothness condition on algebras.

Hochschild cohomology was first introduced for algebras over a field [19], and then extended to algebras over more general rings [10]. Hochschild cohomology groups, especially the ones of lower degrees, have meaningful interpretation, and for this reason, Hochschild cohomology theory has found wide applications in different research fields, such as representation theory, algebraic geometry, mathematical physics and so on.

The calculation of Hochschild cohomology groups is highly nontrivial in general. Some typical well understood examples can be found in [11, 18, 34]. The simplest example next to semi-simple algebras is given by a path algebra over a field, where the Hochschild cohomology groups of higher degrees (≥ 2) vanish, and the dimension of the first Hochschild cohomology group is calculated by counting the number of arrows and paths in the quiver [18].

In many situations, for instance, in the study of the algebraic structures (Gerstenhaber algebra, Batalin-Vilkovisky algebra, and so on) on the Hochschild cohomology groups, it will be very helpful if an explicit basis is given. We mention that even for a path algebra, it is not easy to find in literature such a basis, although every expert working in this area will have one in mind. An interesting observation shows that for any finite acyclic quiver Q, there exists a combinatoric construction for a basis of $\mathrm{HH}^1(\Bbbk Q)$, which is independent of the base field $\Bbbk$; see Proposition 4.6 below.

One may compare the construction with the one in [15], where a basis of the first Hochschild cohomology group of a path algebra has been constructed in a different way in case the quiver is finite planar acyclic and the ground field is of characteristic 0. Recall that a planar quiver is a quiver that can be embedded in the plane.

We simply use $D^b(A)$ to denote the bounded derived category $D^b(A\text{-mod})$ of the category of finitely presented left A-modules. $D^b(A)$ is a triangulated category and one may study its graded center $\mathrm{Z}^*(D^b(A))$. The idea of studying the graded center of a graded category has been used by many authors from different perspectives, see for instance [4, 7, 25, 26, 28].

The derived tensor product induces a map, called the characteristic morphism (a detailed definition will be given in Section 2.3), from the Hochschild cohomology algebra $\mathrm{HH}^*(A)$ of A to $\mathrm{Z}^*(D^b(A))$, which plays an important role in the support theory [2, 4, 5, 31], the deformation theory for abelian and derived categories [27], the block cohomology of groups [25], and in the study of Hochschild cohomology of singular spaces [8].

The characteristic morphism is a homomorphism of graded algebras, which is neither injective nor surjective in general. Naively we may ask when a characteristic morphism will be injective or surjective, and how to characterize the kernel and the image. Only partial answers are known to us at moment.

We give a characterization of the image of an element in $\mathrm{HH}^1(A)$ under the characteristic morphism, which plays an important role in our study of the injectivity of the characteristic morphism, especially its restriction to $\mathrm{HH}^1(A)$.

We only focus on the injectivity problem. We point out that the injectivity condition on a characteristic morphism relates to homological smoothness condition on the algebra closely. Recall that an algebra A is said to be homologically smooth if A has finite projective dimension as A-bimodules. The following result is obtained in Theorem 5.2.

Theorem 1.1 *Let A be a $\Bbbk$-algebra of finite global dimension. If the characteristic morphism of A is injective, then A has finite Hochschild cohomology dimension, i.e., $HH^n(A) = 0$ for sufficiently large n.*

Obviously a homologically smooth algebra has finite Hochschild cohomology dimension, while the finiteness condition on Hochschild cohomology dimension does not imply the homological smoothness in general, and counterexamples can be found in [9]. But it still remains unknown to us whether the injectivity of the characteristic morphism implies the homological smoothness. Inspired by the above theorem, we may ask the following question.

Question Let A be a $\Bbbk$-algebra. Does it hold true that the characteristic morphism of A is injective if and only if A is homologically smooth?

We have only obtained very first results on the question. We begin with the simplest cases. First assume that A has global dimension 0, or equivalently, A is semi-simple. Then the graded center of its derived category is trivial, and we have the following characterization, which is a part of Theorem 5.6 and gives an affirmative answer to the question in semi-simple case.

Theorem 1.2 *Let A be a semi-simple algebra over a field $\Bbbk$. Then the characteristic morphism of A is injective if and only if A is homologically smooth.*

Next we move to hereditary algebras, that is, algebras of global dimension 1. We have an affirmative answer to the question in a specific case, say for finite dimensional path algebras, where there exists a nice basis of its Hochschild cohomology. Note that a path algebra is always homologically smooth, and the above question asks whether the characteristic morphism of a path algebra is injective. The answer is yes. The following result is obtained in Theorem 5.7.

Theorem 1.3 *Let $\Bbbk$ be a field and Q a finite acyclic quiver. Then the characteristic morphism of the path algebra $\Bbbk Q$ is injective.*

We mention that the case of an arbitary hereditary algebra is not known to us yet. The reason is that we do not have a nice description of basis elements in the Hochschild cohomology groups, and our argument does not apply in this case. While if $\Bbbk$ is an algebraically closed field, then we have a more general result: the characteristic morphism of a finite dimensional hereditary $\Bbbk$-algebra is injective. In fact, it was shown in [18] and [29], the Hochschild cohomology is invariant under derived equivalence, see also [21] for a version for differential graded algebras. Now the conclusion follows from the above theorem and the fact that the graded center is a derived invariant.

The paper is organized as follows. In Section 2 we recall some basic notions and notations. Section 3 is devoted to the characterization of the image of a first Hochschild cocycle under the characteristic morphism, and a criterion for the image being zero is also given there.

In Section 4, we construct a basis of the first Hochschild cohomology group of the path algebra of a finite acyclic quiver. We discuss the relation between the injectivity of the characteristic morphism and the homological smoothness condition, and obtain the main results of this paper (Theorem 1.1 through 1.3) in Section 5.

2 Preliminaries

Let A be a $\mathbb{k}$-algebra and M an A-bimodule. Throughout, all unadorned $\otimes$ will denote $\otimes_{\mathbb{k}}$.

2.1 Hochschild Cohomology

Let $\mathrm{HC}^{\bullet}(A, M)$ denote the following complex

$$0 \to \mathrm{Hom}_{\mathbb{k}}(A^0, M) \xrightarrow{\delta^0} \mathrm{Hom}_{\mathbb{k}}(A, M) \xrightarrow{\delta^1} \mathrm{Hom}_{\mathbb{k}}(A^{\otimes 2}, M) \xrightarrow{\delta^2} \cdots$$
$$\xrightarrow{\delta^{n-1}} \mathrm{Hom}_{\mathbb{k}}(A^{\otimes n}, M) \xrightarrow{\delta^n} \mathrm{Hom}_{\mathbb{k}}(A^{\otimes n+1}, M) \xrightarrow{\delta^{n+1}} \cdots$$

of $\mathbb{k}$-spaces, where $A^0 = \mathbb{k}$, and each differential δ^n $(n \geq 1)$ is defined by setting

$$\begin{aligned} &(\delta^n f)(a_1, a_2, \cdots, a_{n+1}) \\ = &\ a_1 f(a_2, \cdots, a_{n+1}) + \textstyle\sum_{i=1}^{n} (-1)^i f(a_1, \cdots, a_i a_{i+1}, \cdots, a_{n+1}) \\ &+ (-1)^{n+1} f(a_1, \cdots, a_n) a_{n+1} \end{aligned}$$

for any $f \in \mathrm{Hom}_{\mathbb{k}}(A^{\otimes n}, M), a_1, a_2, \cdots, a_{n+1} \in A$, and $\delta^0(f)(a) = af(1) - f(1)a$ for any $f \in \mathrm{Hom}_{\mathbb{k}}(\mathbb{k}, M)$, $a \in A$. The n-th Hochschild cohomology group $\mathrm{HH}^n(A, M)$ of A with coefficients in M is defined to be the n-th cohomology group of $\mathrm{HC}^{\bullet}(A, M)$. $\mathrm{HH}^n(A, A)$ is simply denoted by $\mathrm{HH}^n(A)$, and called the n-th Hochschild cohomology group of A.

Remark 2.1 Set $A^e = A \otimes A^{\mathrm{op}}$ to be the enveloping algebra of A. We use m to denote the multiplication map of A. Consider the bar resolution of A

$$\mathrm{Bar}^{\bullet}(A): \qquad \cdots \to A^{\otimes 4} \xrightarrow{d_2} A^{\otimes 3} \xrightarrow{d_1} A^{\otimes 2} \to 0,$$

where for each $n \geq 1$,

$$d_n = \sum_{0 \leq i \leq n} (-1)^i \mathrm{Id}^{\otimes i} \otimes m \otimes \mathrm{Id}^{\otimes n-i}.$$

Together with the multiplication map $m \colon A^{\otimes 2} \to A$, $\mathrm{Bar}^{\bullet}(A)$ gives a projective resolution of A as A-bimodules; and for any A-bimodule M, $\mathrm{Ext}^{\bullet}_{A^e}(A, M)$ is computed as cohomology groups of the complex

$$\cdots \to \mathrm{Hom}_{A^e}(A^{\otimes 4}, M) \to \mathrm{Hom}_{A^e}(A^{\otimes 3}, M) \to \mathrm{Hom}_{A^e}(A^{\otimes 2}, M) \to 0.$$

By applying the natural isomorphism $\mathrm{Hom}_{A^e}(A^{\otimes n+2}, M) \cong \mathrm{Hom}_{\mathbb{k}}(A^{\otimes n}, M)$, the above complex is isomorphic to $\mathrm{HC}^{\bullet}(A, M)$, and consequently we have

$$\mathrm{HH}^n(A, M) = \mathrm{Ext}^n_{A^e}(A, M), n \geq 0.$$

Remark 2.2 (1) The Hochschild cohomology is defined for an algebra A over an arbitrary commutative base ring $\mathbb{k}$, and if A is projective as a $\mathbb{k}$-module, then $\mathrm{HH}^n(A, M) = \mathrm{Ext}^n_{A^e}(A, M)$ for any A-bimodule M and any $n \geq 0$, see Corollary 9.1.5 in [33] for instance.

(2) It was shown in [13] that $\mathrm{HH}^*(A) = \bigoplus_{n \in \mathbb{Z}} \mathrm{HH}^n(A)$ is a positively graded commutative algebra under the Yoneda product; moreover, $\mathrm{HH}^*(A)$ possesses a Gerstenhaber algebra structure.

(3) The lower dimensional Hochschild cohomology groups have the following well-known interpretations:

(a) $\mathrm{HH}^0(A) \cong \mathrm{Z}(A)$, where $\mathrm{Z}(A)$ denotes the center of A;

(b) $\mathrm{HH}^1(A) \cong \mathrm{Der}(A)/\mathrm{InDer}(A)$, where $\mathrm{Der}(A)$ is the $\Bbbk$-space of derivations of A, and $\mathrm{InDer}(A)$ the subspace consisting of inner ones.

Recall that a derivation of A is a linear map $\delta \in \mathrm{Hom}_{\Bbbk}(A, A)$ with $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in A$. For each $a \in A$, the map $\partial_a = [a, -]$, $x \mapsto ax - xa$ defines a derivation, and we call it the inner derivation induced by a. Clearly, inner derivations form a $\Bbbk$-subspace. If a derivation is not inner, then we call it an outer derivation.

(c) $\mathrm{HH}^2(A)$ controls the infinitesimal deformations of A, and $\mathrm{HH}^3(A)$ describes the obstruction to extending an infinitesimal deformation to a formal one. We refer to [14] for an explanation.

2.2 Graded Center of a Triangulated Category

By a graded $\Bbbk$-category we mean a pair $(\mathcal{C}, \Sigma)$, where $\mathcal{C}$ is a $\Bbbk$-category and Σ an auto-equivalence of $\mathcal{C}$. The graded center of a graded $\Bbbk$-category $(\mathcal{C}, \Sigma)$, denoted by $\mathrm{Z}^*(\mathcal{C}, \Sigma)$, is defined to be the $\mathbb{Z}$-graded $\Bbbk$-space $\mathrm{Z}^*(\mathcal{C}, \Sigma) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Z}^n(\mathcal{C}, \Sigma)$, where for each n,

$$\mathrm{Z}^n(\mathcal{C}, \Sigma) = \{\eta \colon \mathrm{Id}_{\mathcal{C}} \to \Sigma^n \mid \Sigma\eta = (-1)^n \eta\Sigma\}.$$

For any $\eta \in \mathrm{Z}^n(\mathcal{C}, \Sigma)$ and $\zeta \in \mathrm{Z}^m(\mathcal{C}, \Sigma)$, a routine check shows that $\Sigma^m\eta \circ \zeta \in \mathrm{Z}^{m+n}(\mathcal{C}, \Sigma)$. Thus the assignment $\eta\zeta = \Sigma^m\eta \circ \zeta$ defines a multiplication on $\mathrm{Z}^*(\mathcal{C}, \Sigma)$, which is easily shown to be associative and graded commutative, where graded commutative means that $\eta\zeta = (-1)^{mn}\zeta\eta$ for any $\eta \in \mathrm{Z}^n(\mathcal{C}, \Sigma)$ and $\zeta \in \mathrm{Z}^m(\mathcal{C}, \Sigma)$.

Remark 2.3 (1) The classical center $\mathrm{Z}(\mathcal{C})$ of a category $\mathcal{C}$ is formed by all natural transformations from the identity functor to itself. Clearly, $\mathrm{Z}^0(\mathcal{C}, \Sigma) \subseteq \mathrm{Z}(\mathcal{C})$, and the equality holds if $\Sigma = \mathrm{Id}_{\mathcal{C}}$.

(2) Note that $\mathrm{Z}^*(\mathcal{C}, \Sigma)$ is not necessarily a set in general; it will be if $\mathcal{C}$ is small.

A triangulated category is a graded category $(\mathcal{C}, \Sigma)$ together with a class of triangles, called distinguished triangles, satisfying certain axioms. Recall that a tringle (X, Y, Z, f, g, h), usually written as $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$, consists of three objects X, Y and Z together with morphisms $f \colon X \to Y$, $g \colon Y \to Z$ and $h \colon Z \to \Sigma X$. In this case, the graded center $\mathrm{Z}^*(\mathcal{C}, \Sigma)$ is called the graded center of the triangulated category $\mathcal{C}$, and simply denoted by $\mathrm{Z}^*(\mathcal{C})$. We refer to [17, 32] for details on triangulated categories.

The condition $\Sigma\eta = (-1)^n\eta\Sigma$ in the definition seems to be meaningful in this case. In fact, for any $\eta \in \mathrm{Z}^n(\mathcal{C})$ and any distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ in $\mathcal{C}$, the equality $\Sigma\eta = (-1)^n\eta\Sigma$ ensures the existence of a morphism of triangles

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\
\downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle (-1)^n\eta_Y} & & \downarrow{\scriptstyle \eta_Z} & & \downarrow{\scriptstyle \Sigma\eta_X} \\
\Sigma^n X & \xrightarrow{(-1)^n\Sigma^n f} & \Sigma^n Y & \xrightarrow{(-1)^n\Sigma^n g} & \Sigma^n Z & \xrightarrow{(-1)^n\Sigma^n h} & \Sigma^{n+1} X.
\end{array}$$

2.3 The Characteristic Morphism

Let A be an algebra. We denote by A-Mod the category of left A-modules, and A-mod the full subcategory of finitely presented ones. We simply denote $D(A\text{-Mod})$ by $D(A)$ and $D^b(A\text{-mod})$ by $D^b(A)$. Note that A-mod is a small exact category; and it will be an abelian category if A is left coherent. Moreover, by Lemma 7.7 in [22], the embedding $A\text{-mod} \subseteq A\text{-Mod}$ induces a fully faithful exact functor $D^b(A) \to D^b(A\text{-Mod})$.

It is well known that $D^b(A)$ is a triangulated category with the shift functor [1], and the category A-mod is a full subcategory of $D^b(A)$, where each A-module is viewed as a stalk complex concentrated in degree 0. Moreover, for any pair of A-modules M and N, we have natural isomorphisms

$$\mathrm{Ext}_A^n(M, N) \cong \mathrm{Hom}_{D(A)}(M, N[n]), n \geq 0.$$

For each $n \geq 0$, applying the natural isomorphisms $\mathrm{HH}^n(A) = \mathrm{Ext}^n_{A^e}(A, A)$ and $\mathrm{Ext}^n_{A^e}(A, A) \cong \mathrm{Hom}_{D(A^e)}(A, A[n])$, any $\eta \in \mathrm{HH}^n(A)$ can be viewed as a morphism in $\mathrm{Hom}_{D(A^e)}(A, A[n])$, which is again denoted by η. Clearly η induces a natural transformation $\chi(\eta)\colon A \overset{\mathbb{L}}{\otimes}_A - \longrightarrow A[n] \overset{\mathbb{L}}{\otimes}_A -$, where

$$A \overset{\mathbb{L}}{\otimes}_A -, A[n] \overset{\mathbb{L}}{\otimes}_A -\colon D(A) \to D(A)$$

are left derived functors of $A \otimes_A -$ and $A[n] \otimes_A -$ respectively. Using the fact A is flat as a right A-module, we obtain natural isomorphisms $A \overset{\mathbb{L}}{\otimes}_A - = \mathrm{Id}_{D(A)}$, and $A[n] \overset{\mathbb{L}}{\otimes}_A - = [n]$.

It is routine to check that $\chi(\eta) \in \mathrm{Z}^n(D(A))$. Thus there is a map from $\mathrm{HH}^*(A)$ to $\mathrm{Z}^*(D^b(A))$. Composed with the restriction map $\mathrm{Z}^*(D(A)) \to \mathrm{Z}^*(D^b(A))$, we obtain a well-defined map $\chi : \mathrm{HH}^*(A) \to \mathrm{Z}^*(D^b(A))$. Here we use the fact that $D^b(A)$ is a full subcategory of $D(A)$.

Remark 2.4 It is shown in [28, 4.5] that χ is a homomorphism of graded commutative $\Bbbk$-algebras. In fact, in geometric contexts such a map can be viewed as an algebraic version of the Fourier-Mukai transformation, see [7, 3.3]. χ is called the characteristic morphism of A.

We mention that the above construction also defines homomorphisms $\mathrm{HH}^*(A) \to \mathrm{Z}^*(D(A))$ and $\mathrm{HH}^*(A) \to \mathrm{Z}^*(D^b(A\text{-Mod}))$, and χ factors through both of them.

Remark 2.5 Obviously, for an arbitrary abelian category $\mathcal{C}$, any endomorphism of $\mathrm{Id}_{\mathcal{C}}$ lifts to an endomorphism of $\mathrm{Id}_{D(\mathcal{C})}$ which commutes with the shift functor [1], where $D(\mathcal{C})$ denotes the derived category of $\mathcal{C}$, bounded or unbounded. In other words, there is a natural embedding of graded rings $\mathrm{Z}(\mathcal{C}) \hookrightarrow \mathrm{Z}^0(D(\mathcal{C}))$.

We apply this fact to module categories. A classical result says that there are isomorphisms $\mathrm{Z}(A) \cong \mathrm{Z}(A\text{-mod}) \cong \mathrm{Z}(A\text{-Mod})$ of commutative $\Bbbk$-algebras. Thus we obtain embeddings $\mathrm{Z}(A) \hookrightarrow \mathrm{Z}^0(D(A))$ and $\mathrm{Z}(A) \hookrightarrow \mathrm{Z}^0(D^b(A))$. We may identify $\mathrm{HH}^0(A)$ with $\mathrm{Z}(A)$, and the above inclusions are compatible with

$$\chi^0\colon \mathrm{HH}^0(A) \to \mathrm{Z}^0(D(A)) \to \mathrm{Z}^0(D^b(A)),$$

the degree 0 component of the characteristic morphism.

3 The Natural Transformation Induced by a Derivation

3.1 Derivations and Bimodule Extensions

We first introduce some useful notations. Let A be an algebra and $\delta \in \mathrm{Der}(A)$ a derivation of A. For any left A-module M, $M \oplus_\delta M$ is defined to be the A-module with the underlying $\Bbbk$-space $M \oplus M$, and the module action given by $a(m, n) = (am + \delta(a)n, an)$ for any $a \in A$ and $m, n \in M$. In particular, $A \oplus_\delta A$ is an A-bimodule under the usual right A-module action.

Remark 3.1 (1) If δ, δ' are derivations of A with $\delta - \delta' \in \mathrm{InDer}(A)$, then there exists an isomorphism of A-bimodules $A \oplus_\delta A \cong A \oplus_{\delta'} A$. In fact, assume $\delta - \delta' = \partial_a$ for some $a \in A$, then the map $F: A \oplus_\delta A \to A \oplus_{\delta'} A$, $(x, y) \mapsto (x + ay, y)$ gives the desired isomorphism. In particular, if δ is an inner derivation, then $A \oplus_\delta A \cong A \oplus A$, the usual direct sum.

(2) It is easy to check that $M \oplus_\delta M \cong (A \oplus_\delta A) \otimes_A M$.

(3) We have the following general construction. Let δ be a derivation of A and $f: N \to M$ a homomorphism of left A-modules. Then we may endow the $\Bbbk$-space $M \oplus N$ a left A-module structure with the module action given by $a(m, n) = (am + \delta(a)f(n), an)$ for any $m \in M$, $n \in N$, and $a \in A$. Then $M \oplus_\delta M$ is the special case by setting $M = N$ and $f = \mathrm{Id}_M$.

For any A-modules M, N, we denote by $\mathrm{Yext}^1(M, N)$ the set of isoclasses of short exact sequences of A-modules with the first term N and the last term M. There exists a natural isomorphism of bi-functors $\mathrm{Yext}^1_A(-, -) \cong \mathrm{Ext}^1_A(-, -)$. The following lemma explains why we introduce the above construction.

Lemma 3.2 *Under the natural isomorphisms $HH^1(A) \cong Ext^1_{A^e}(A, A) \cong \mathrm{Yext}^1_{A^e}(A, A)$, a derivation δ of A maps to the isoclass of the short exact sequence*

$$0 \to A \xrightarrow{\binom{1}{0}} A \oplus_\delta A \xrightarrow{(0,1)} A \to 0.$$

Proof The assertion follows directly from the commutative diagram of homomorphisms of A-bimodules

$$\begin{array}{ccccccccc}
A^{\otimes 4} & \xrightarrow{d_2} & A^{\otimes 3} & \xrightarrow{d_1} & A^{\otimes 2} & \xrightarrow{m} & A & \longrightarrow & 0 \\
 & & \downarrow f & & \downarrow g & & \| & & \\
0 & \longrightarrow & A & \xrightarrow{\binom{1}{0}} & A \oplus_\delta A & \xrightarrow{(0,1)} & A & \longrightarrow & 0,
\end{array}$$

where f, g are given by $f(a \otimes b \otimes c) = a\delta(b)c$ and $g(a \otimes b) = (\delta(a)b, ab)$. □

3.2 Derivations Viewed as Morphisms in Derived Category

As we have mentioned above, there is a natural isomorphism $\mathrm{Ext}^1_A(M, N) \cong \mathrm{Hom}_{D(A)}(M, N[1])$ for any A-modules M, N. Note that each A-module M can be viewed as a stalk complex concentrated in degree 0, which by abuse of notation also denoted by M.

Let ξ be a short exact sequence $0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0$. We denote by $[\xi]$ the isoclass of short exact sequences which contains ξ. Denote by $X^\bullet_\xi$ the complex

$$\cdots \longrightarrow 0 \longrightarrow N \xrightarrow{f} E \longrightarrow 0 \longrightarrow \cdots,$$

where $X^0_\xi = E$, $X^1_\xi = N$ and $X^i_\xi = 0$ for $i \neq 0, 1$. Then we have a quasi-isomorphism $s: X^\bullet_\xi \Longrightarrow M$, say

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & N & \xrightarrow{f} & E & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow g & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & \cdots.
\end{array}$$

Here we use the usual notation " $\Longrightarrow$ " to denote a quasi-isomorphism of complexes. Moreover, we have a morphism of complexes $u\colon X_\xi^\bullet \to N[1]$ given by

$$\begin{CD} \cdots @>>> 0 @>>> N @>f>> E @>>> 0 @>>> \cdots \\ @. @VVV @| @VVV @VVV @. \\ \cdots @>>> 0 @>>> N @>>> 0 @>>> 0 @>>> \cdots , \end{CD}$$

where the bottom complex is $N[1]$, a stalk complex concentrated in degree 1. Now $M \overset{s}{\Longleftarrow} X_\xi^\bullet \overset{u}{\longrightarrow} N[1]$ (usually called a roof) gives a morphism $u/s\colon M \to N[1]$ in $D(A)$. Note that u/s can be viewed as $u \circ s^{-1}$.

We mention that the map $u \circ s^{-1}$ is independent of the choice of representatives in $[\xi]$. In fact, let $\xi' : 0 \to N \overset{f'}{\to} E' \overset{g'}{\to} M \to 0$ be a short exact sequence isomorphic to ξ. Then we have a commutative diagram

$$\begin{CD} 0 @>>> N @>f>> E @>g>> M @>>> 0 \\ @. @| @VV{\alpha}V @| @. \\ 0 @>>> N @>{f'}>> E' @>{g'}>> M @>>> 0, \end{CD}$$

which induces an isomorphism of complex $\tilde{\alpha}\colon X_\xi^\bullet \to X_{\xi'}^\bullet$ given by

$$\begin{CD} 0 @>>> N @>f>> E @>>> 0 \\ @. @| @VV{\alpha}V @. \\ 0 @>>> N @>{f'}>> E' @>>> 0. \end{CD}$$

We may define $s'\colon X_{\xi'}^\bullet \to M$ and $u'\colon X_{\xi'}^\bullet \to N[1]$ similarly. Clearly we have $s = s' \circ \tilde{\alpha}$ and $u = u' \circ \tilde{\alpha}$, and hence $u/s = u/s'$.

We have shown that the above assignment defines a map from $\mathrm{Yext}_A^1(M, N)$ to $\mathrm{Hom}_{D(A)}(M, N[1])$, which coincides with the composite of the isomorphisms

$$\mathrm{Yext}_A^1(M, N) \cong \mathrm{Ext}_A^1(M, N) \cong \mathrm{Hom}_{D(A)}(M, N[1]).$$

In particular, for a derivation $\delta \in \mathrm{Der}(A)$, the image of δ in $\mathrm{Hom}_{D(A^e)}(A, A[1])$ is shown as follows.

$$\begin{CD} \cdots @>>> 0 @>>> 0 @>>> A @>>> 0 @>>> \cdots \\ @. @AAA @AAA @AA{(0,1)}A @AAA @. \\ \cdots @>>> 0 @>>> A @>{\binom{1}{0}}>> A \oplus_\delta A @>>> 0 @>>> \cdots \\ @. @VVV @| @VVV @VVV @. \\ \cdots @>>> 0 @>>> A @>>> 0 @>>> 0 @>>> \cdots \end{CD}$$

3.3 The Image of a Derivation

We are now ready to study the image of a derivation under the characteristic morphism. Let δ be a derivation of A, and $\bar{\delta}$ its image in $\mathrm{HH}^1(A)$. By definition, when restricted to an A-module M, the map $\chi(\bar{\delta})_M : M \to M[1]$ is given as follows.

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow{\scriptstyle (0,1)} & & \uparrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & M & \xrightarrow{\binom{1}{0}} & M \oplus_\delta M & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \| & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

Note that we use the fact $(A \oplus_\delta A) \otimes_A M \cong M \oplus_\delta M$ here. Then we have the following useful lemma.

Lemma 3.3 *Let δ be a derivation of A. Then for any A-module M, under the isomorphisms $Hom_{D(A)}(M, M[1]) \cong Ext^1_A(M, M) \cong \mathrm{Yext}^1_A(M, M)$, the map $\chi(\bar{\delta})_M : M \to M[1]$ maps to the isoclass of the short exact sequence*

$$0 \to M \xrightarrow{\binom{1}{0}} M \oplus_\delta M \xrightarrow{(0,1)} M \to 0.$$

We draw the following easy consequence.

Proposition 3.4 *Let δ be a derivation of A. If $\chi(\bar{\delta}) = 0$, then the short exact sequence $0 \to M \xrightarrow{\binom{1}{0}} M \oplus_\delta M \xrightarrow{(0,1)} M \to 0$ splits for any A-module M. Moreover, if A is hereditary, then the converse statement also holds true.*

Proof Note that for an A-module M, $0 \to M \xrightarrow{\binom{1}{0}} M \oplus_\delta M \xrightarrow{(0,1)} M \to 0$ splits if and only if $\chi(\bar{\delta})_M = 0$. Thus the first assertion is obvious.

A classical result says that for a hereditary algebra A, any object in $D(A)$ can be written as a direct sum of stalk complexes. If $\chi(\bar{\delta})_M = 0$ for any stalk complex M concentrated in degree zero, then $\chi(\bar{\delta})_M = 0$ holds for all stalk complexes and hence for all complexes. The last conclusion follows. □

We end this section with two technical lemmas for later use.

Lemma 3.5 *Let δ be a derivation of A and M an A-module. Then the sequence $0 \to M \xrightarrow{\binom{1}{0}} M \oplus_\delta M \xrightarrow{(0,1)} M \to 0$ splits if and only if there exists a $\Bbbk$-linear map $f : M \to M$, such that $f(am) = af(m) + \delta(a)m$ for any $a \in A$ and $m \in M$.*

The proof is easy. In fact, $0 \to M \xrightarrow{\binom{1}{0}} M \oplus_\delta M \xrightarrow{(0,1)} M \to 0$ splits if and only if there exists an A-module homomorphism $\binom{f}{g} : M \to M \oplus_\delta M$ such that $(0, 1)\binom{f}{g} = \mathrm{Id}_M$. The last equality implies that $g = 1$; and $\binom{f}{1}$ is an A-module homomorphism if and only if $f(am) = af(m) + \delta(a)m$.

Lemma 3.6 *Let δ be a derivation of A. Suppose that the sequence*

$$0 \to M \xrightarrow{\binom{1}{0}} M \oplus_\delta M \xrightarrow{(0,1)} M \to 0$$

splits for any A-module M. Then $\delta(a) \in \langle a\rangle$ for any $a \in A$, where $\langle a\rangle = AaA$ is the ideal generated by a.

Proof We consider the module $M = A/\langle a\rangle$. By Lemma 3.5, there exists some $f : M \to M$, such that for any $m \in M$, $f(am) = af(m) + \delta(a)m$. Thus $0 = f(\bar{a}) = f(a\bar{1}) = af(\bar{1}) + \delta(a)\bar{1} = \overline{\delta(a)}$, which is equivalent to $\delta(a) \in \langle a\rangle$. □

4 Derivations of a Path Algebra

In this section we will give a basis of the Hochschild cohomology group of a finite dimensional path algebra, which is handy in our later use. We mention that such a basis should be known to people working in this area, although it is difficult to find it in literature.

4.1 Quivers and Path Algebras

A quiver is, roughly speaking, an oriented graph. More precisely, a quiver $Q = (Q_0, Q_1, s, t)$ is given by a quadruple, where Q_0 is the set of vertices, Q_1 the set of edges which are usually called arrows, and $s, t : Q_1 \to Q_0$ are two maps assigning to each arrow α its starting vertex and terminating vertex respectively. If both Q_0 and Q_1 are finite sets, then we call Q a finite quiver. All quivers considered here are assumed to be finite.

A non-trivial path p in Q is a sequence of arrows $\alpha_n \cdots \alpha_2\alpha_1$ with $t(\alpha_i) = s(\alpha_{i+1})$ for $1 \le i \le n-1$; $s(\alpha_1)$ and $t(\alpha_n)$ are called the starting vertex and terminating vertex of p, and denoted by $s(p)$ and $t(p)$ respectively; and n is called the length of p, and denoted by $l(p)$. For each vertex v, we denote by e_v the trivial path which starts and terminates at v and which has length 0. The set of all paths in Q is also denoted by Q.

Set $\Bbbk Q$ be the $\Bbbk$-space with a basis consisting of all paths in Q. Then the concatenation of paths defines an associative multiplication on $\Bbbk Q$. We call such an algebra the path algebra of Q and denote it by $\Bbbk Q$. $\Bbbk Q$ has an identity element if and only if Q_0 is a finite set, and in this case, $1_{\Bbbk Q} = \sum_{v \in Q_0} e_v$. For unexplained notions on quivers and path algebras, we refer to [1] and [30].

Remark 4.1 (1) An oriented cycle in Q means a nontrivial path with the same starting and terminating vertex. A quiver Q is said to be acyclic if Q has no oriented cycles. A path algebra $\Bbbk Q$ is finite dimensional if and only if Q is a finite acyclic quiver.
(2) The path algebra $\Bbbk Q$ is a hereditary algebra. If $\Bbbk$ is an algebraically closed field, then any finite dimensional $\Bbbk$-algebra is Morita equivalent to a path algebra of some finite

quiver modulo an admissible ideal, in particular, any finite dimensional hereditary $\mathbb{k}$-algebra is Morita equivalent to a path algebra of some finite acyclic quiver.

(3) It was also shown in [18] that $\mathrm{HH}^n(\mathbb{k}Q) = 0$ for any finite quiver Q and any $n \geq 2$. Moreover, if Q is connected and is not an oriented cycle, then $\mathrm{HH}^0(\mathbb{k}Q) = \mathbb{k}$.

4.2 Derivations of a Path Algebra

Let Q be a finite quiver and $A = \mathbb{k}Q$ the path algebra. We say that two paths p, p' in Q are parallel, denoted by $p \parallel p'$, if $s(p) = s(p')$ and $t(p) = t(p')$. For any vertices $s, t \in Q_0$, $Q_{t,s}$ denotes the set of all paths starting at s and terminating at t. Clearly, $1 = \sum_{v \in Q_0} e_v$ is an orthogonal idempotents decomposition of the identity, that is, $e_s e_t = \delta_{s,t} e_s$ for all $s, t \in Q_0$, here we use the Kronecker delta notation. The idempotent decomposition gives a double Pierce decomposition $A = \bigoplus_{s,t \in Q_0} e_t A e_s$. Note that each $e_t A e_s$ has a $\mathbb{k}$-basis $Q_{t,s}$.

We have the following well-known results on derivations of a path algebra. For the convenience of the readers we also include a proof. We mention that the result holds true for any associative algebra and any orthogonal idempotent decomposition of the identity.

Lemma 4.2 *Let d be a derivation of A. Then there exists a derivation $\tilde{d}$ such that $\tilde{d} - d \in$ $\mathrm{InDer}(A)$ and $\tilde{d}(e_v) = 0$ for any vertex $v \in Q_0$.*

Proof It is equivalent to show that there exists some inner derivation d' such that $d'(e_v) = d(e_v)$ holds for all $v \in Q_0$.

For any $v \in Q_0$, we may write $d(e_v) = \sum_{s,t \in Q_0} X^v_{t,s}$, where $X^v_{t,s} \in \mathbb{k}Q_{t,s}$ are uniquely determined by d.

Since d is a derivation and e_v an idempotent, we have

$$d(e_v) = d(e_v^2) = e_v d(e_v) + d(e_v) e_v,$$

and hence

$$\sum_{s,t \in Q_0} X^v_{t,s} = \sum_{s \in Q_0} X^v_{v,s} + \sum_{t \in Q_0} X^v_{t,v} = \sum_{s \in Q_0, s \neq v} X^v_{v,s} + \sum_{t \in Q_0, t \neq v} X^v_{t,v} + 2X^v_{v,v}.$$

By comparing the left side and the right side of the above equality, we obtain that $X^v_{v,v} = 0$, and $X^v_{s,t} = 0$ if none of s and t equals to v.

Moreover, by applying $d(1) = 0$, we know that $X^s_{t,s} = -X^t_{t,s}$ holds for any $s \neq t \in Q_0$. Set $a = \sum_{s,t \in Q_0} X^s_{t,s}$, and let ∂_a be the induced inner derivation. Then for any $v \in Q_0$, we have

$$\begin{aligned} \partial_a(e_v) &= \sum_{s,t \in Q_0} X^s_{t,s} e_v - \sum_{s,t \in Q_0} e_v X^s_{t,s} \\ &= \sum_{t \in Q_0} X^v_{t,v} e_v - \sum_{s \in Q_0} e_v X^s_{v,s} \\ &= \sum_{t \in Q_0} X^v_{t,v} e_v + \sum_{s \in Q_0} e_v X^v_{v,s} = d(e_v), \end{aligned}$$

which completes the proof. □

Lemma 4.3 *Let d be a derivation of A with $d(e_v) = 0$ for any $v \in Q_0$. Then for any nontrivial path p, $d(p)$ is a linear combination of paths parallel to p.*

Proof Set $s = s(p)$ and $t = t(p)$. Then

$$d(p) = d(e_t p e_s) = d(e_t) p e_s + e_t d(p) e_s + e_t p d(e_s) = e_t d(p) e_s,$$

and the conclusion follows. □

4.3 A Basis of $\mathrm{HH}^1(\Bbbk Q)$ for a Finite Acyclic Quiver Q

As an algebra the path algebra $A = \Bbbk Q$ is generated by $\Bbbk Q_0$ and $\Bbbk Q_1$, therefore by the Leibniz rule any derivation of A is determined by its values on all trivial paths and arrows. To give an explicit basis of $\mathrm{HH}^1(\Bbbk Q)$, we need the following notation.

Let $\alpha \in Q_0$ be an arrow in Q_1 and $p \parallel \alpha$ a (possibly trivial) path in Q parallel to α. Then there exists a unique derivation of A, denoted by ∂_α^p, such that $\partial_\alpha^p(e_v) = 0$ for any $v \in Q_0$, $\partial_\alpha^p(\alpha) = p$, and $\partial_\alpha^p(x) = 0$ for any $x \in Q_1$ with $x \neq \alpha$.

Remark 4.4 By Lemma 4.2, the set $\{\partial_\alpha^p \mid \alpha \in Q_0, p \in Q, p \parallel \alpha\}$ together with $\mathrm{InDer}(A)$ linearly span $\mathrm{Der}(A)$. In other words, $\mathrm{HH}^1(A) = \mathrm{Der}(A)/\mathrm{InDer}(A)$ is linearly spanned by the image of all ∂_α^p.

As usual, we also denote the image of a derivation d of A in $\mathrm{HH}^1(A)$ by $\bar{d}$. We mention that ∂_α^p's are linearly independent in $\mathrm{Der}(A)$, while $\bar{\partial}_\alpha^p$'s are not in $\mathrm{HH}^1(A)$. To obtain a basis of $\mathrm{HH}^1(A)$, it suffices to remove some redundant derivations.

Recall that a spanning forest Γ of a quiver Q is a subquiver of Q whose underlying graph contains no cycles and which is maximal with this property.

Remark 4.5 (1) Spanning forests are not unique in general.

(2) Assume Q has r connected components. Then the maximality requirement implies that $|\Gamma_1| = |Q_0| - r$, and the number of connected components of Γ is also r. Moreover, two vertices v_1 and v_2 belong to the same connected component of Γ if and only if they belong to the same connected component of Q. In particular, if Q is connected, then Γ is connected and has exactly $|Q_0| - 1$ edges.

Proposition 4.6 *Let Q be a finite acyclic quiver, $A = \Bbbk Q$ the path algebra, and Γ a spanning forest of Q. Then*

$$\left\{\bar{\partial}_\alpha^p \mid \alpha \in Q_1, p \in Q, p \parallel \alpha\} \setminus \{\bar{\partial}_\alpha^\alpha | \alpha \in \Gamma_1\right\}$$

is a basis of $HH^1(A) = \mathrm{Der}(A)/\mathrm{InDer}(A)$.

Proof Set $X = \{\partial_\alpha^p \mid \alpha \in Q_1, p \in Q, p \parallel \alpha\}$ and $X_\Gamma = X \setminus \{\partial_\alpha^\alpha | \alpha \in \Gamma_1\}$, and denote by $\bar{X}$, $\bar{X}_\Gamma$ their images in $\mathrm{HH}^1(A)$. To show that $\bar{X}_\Gamma$ gives a basis, one needs to show that derivations in $\bar{X}_\Gamma$ are linearly independent in $\mathrm{HH}^1(A)$, and $\bar{X}_\Gamma$ linearly generates $\mathrm{HH}^1(A)$.

Firstly, suppose that $a_\alpha, b_{\beta,p} \in \Bbbk$ are such that

$$\sum_{\alpha \in Q_1 \setminus \Gamma_1} a_\alpha \partial_\alpha^\alpha + \sum_{\substack{\beta \in Q_1, p \in Q \\ p \parallel \beta, p \neq \beta}} b_{\beta,p} \partial_\beta^p \in \mathrm{InDer}(A),$$

that is,

$$\sum_{\alpha\in Q_1\backslash\Gamma_1} a_\alpha \partial_\alpha^\alpha + \sum_{\substack{\beta\in Q_1,p\in Q\\ p\|\beta,p\neq\beta}} b_{\beta,p}\partial_\beta^p = \sum_{v\in Q_0} c_v\partial_{e_v} + \sum_{p\in Q_{\geq 1}} d_p\partial_p \tag{4.1}$$

for some $c_v, d_p \in \mathbb{k}$, here $Q_{\geq 1}$ denotes the set of all non-trivial paths.

Obviously by evaluating the Eq. 4.1 at any $e_v (v \in Q_0)$, we obtain that

$$\sum_{p\in Q_{\geq 1},s(p)=v} d_p p - \sum_{p\in Q_{\geq 1},t(p)=v} d_p p = 0.$$

By assumption Q is acyclic, which implies that for any nontrivial path p, $s(p) = v$ and $t(p) = v$ never occur simultaneously, then the above equality forces $d_p = 0$ whenever $s(p) = v$ or $t(p) = v$, and hence all d_p's vanish.

Moreover, for any $\beta \in Q_1$, by evaluating the Eq. 4.1 at β we have $b_{\beta,p} = 0$ for all $p \neq \beta$, $p \parallel \beta$. Thus (4.1) reads as

$$\sum_{\alpha\in Q_1\backslash\Gamma_1} a_\alpha\partial_\alpha^\alpha = \sum_{v\in Q_0} c_v\partial_{e_v}.$$

Now for any arrow $\gamma \in \Gamma_1 \subseteq Q_1$, by evaluating the above equation at γ, we have

$$0 = \sum_{\alpha\in Q_1\backslash\Gamma_1} a_\alpha\partial_\alpha^\alpha(\gamma) = \sum_{v\in Q_0} c_v\partial_{e_v}(\gamma) = (c_{t(\gamma)} - c_{s(\gamma)})\gamma,$$

and hence $c_{t(\gamma)} = c_{s(\gamma)}$. It forces that all c_v's with v in the same connected component of Γ are equal. Since Γ is a spanning forest of Q, we deduce that all c_v's with v in the same connected component of Q are equal. For any connected component Q' of Q, it is easy to check that $\sum_{v\in Q_0'}\partial_{e_v} = 0$, and it follows that $\sum_{\alpha\in Q_1\backslash\Gamma_1} a_\alpha\partial_\alpha^\alpha = \sum_{v\in Q_0} c_v\partial_{e_v} = 0$. Thus given any $\gamma \in Q_1 \setminus \Gamma_1$, we have $a_\gamma\gamma = \sum_{\alpha\in Q_1\backslash\Gamma_1} a_\alpha\partial_\alpha^\alpha(\gamma) = 0$, and hence $a_\gamma = 0$.

We have shown that all the coefficients a_α's and $b_{\beta,p}$'s must be 0, and the linear independence of $\bar{X}_\Gamma$ follows.

Next we show that X_Γ generates $\mathrm{HH}^1(A)$. Since $\bar{X}$ spans $\mathrm{HH}^1(A)$, it suffices to show that any ∂_α^α with $\alpha \in \Gamma_1$ is a linear combination of elements in X_Γ and inner derivations.

For any $\alpha \in \Gamma_1$, consider the subquiver $\Gamma_\alpha = \Gamma \setminus \alpha$. Now we may define a $\{0, 1\}$-valued function $\mathbf{f}_\alpha$ on Q_0 as follows,

$$\mathbf{f}_\alpha(v) = \begin{cases} 1, & \text{if } v \text{ and } t(\alpha) \text{ belong to the same connected component of } \Gamma_\alpha; \\ 0, & \text{otherwise.} \end{cases}$$

Then it is straightforward to verify that

$$\partial_\alpha^\alpha = \sum_{e\in Q_0} \mathbf{f}_\alpha(v)\partial_{e_v} - \sum_{\beta\in Q_1\backslash\Gamma_1} (\mathbf{f}_\alpha(t(\beta)) - \mathbf{f}_\alpha(s(\beta)))\partial_\beta^\beta,$$

which completes the proof. □

We draw the following easy consequence.

Corollary 4.7 *([16]) Let Q be a finite connected acyclic quiver. For any $\alpha \in Q_1$, denote by $v(\alpha)$ the number of paths parallel to α. Then*

$$\dim_k HH^1(A) = \sum_{\alpha\in Q_1} v(\alpha) - (|Q_0| - 1).$$

5 Characteristic Morphism of a Path Algebra

The characteristic morphism of an algebra is neither injective nor surjective in general. In this section, we will discuss the question when a characteristic morphism is injective or surjective, and show how the injectivity of the characteristic morphism relates to certain smoothness condition closely.

5.1 We Begin with an Interesting Example in which the Characteristic Morphism is Neither Injective Nor Surjective

Example 5.1 Let $\Bbbk$ be a field of characteristic 0, and $A = \Bbbk[x]/\langle x^2\rangle$ the ring of dual numbers over $\Bbbk$. It is known that $\mathrm{HH}^1(A) \cong \Bbbk$. By some tedious calculation one shows that the characteristic morphism χ maps $\mathrm{HH}^1(A)$ to 0. Thus χ is not injective in this case.

Moreover, $\mathrm{Z}^0(D^b(A))$ is of infinite dimension, thus the restriction of χ to $\mathrm{HH}^0(A)$ $(= A)$ is not surjective, see [24] or [23, Propostion 5.2].

It has been implicitly shown in [24, Section 2] that for a triangulated category satisfying certain Krull-Schmidt property, for instance the bounded derived category of a finite dimensional algebra of infinite representation type, the graded center could be of uncountable dimension, see also [23, Section 4] for tame hereditary algebra case. Since each component of the Hochschild cohomology group of a finite dimensional algebra is always finite dimensional, the characteristic morphism is not surjective in this case.

We mainly discuss finite dimensional algebras here, and we only consider the injectivity of characteristic morphisms. We first show that for an algebra A (not necessarily of finite dimension) with finite global dimension the injectivity of the characteristic morphism implies the finiteness of the Hochschild cohomology dimension of A, which is by definition the least integer n such that $\mathrm{HH}^n(A) \neq 0$, see [16, Definition 2].

Theorem 5.2 *Let A be a $\Bbbk$-algebra of finite global dimension. If the characteristic morphism of A is injective, then $HH^n(A) = 0$ for sufficiently large n.*

Recall that an algebra A is said to be homologically smooth if A has a bounded resolution of projective A-bimodules, or equivalently, the projective dimension of A as an A-bimodule is finite. A homologically smooth algebra always has finite Hochschild cohomology dimension. The converse is not true in general. In fact, there exists an algebra A of finite Hochschild cohomology dimension but of infinite global dimension, see for instance [9]. Combined with the fact $\mathrm{proj.dim}_{A^e} A \geq \mathrm{gl.dim} A$, we know that A is not homologically smooth. But it still remains unknown to us whether the injectivity of the characteristic morphism implies the homological smoothness.

The theorem is an easy consequence of the following proposition. Note that the characteristic morphism factors through $\mathrm{Z}^m(D^b(A\text{-Mod}))$.

Proposition 5.3 *Assume A has global dimension d. Then* $\mathrm{Z}^m(D^b(A\text{-Mod})) = 0$ *unless* $0 \leq m \leq d$.

To simplify the proof, we need some notation. Let $X^\bullet$ be a complex of A-modules and $n \in \mathbb{Z}$ an integer. We denote by $\iota_n X^\bullet$ the truncated complex with $(\iota_n X^\bullet)^m = 0$ for $m \geq$

$n+1$, and $(\iota_n X^\bullet)^m = X^m$ for $m \le n$, and $d^m_{\iota_n X^\bullet} = d^m_{X^\bullet}$ for $m \le n$. Thus there exists a natural map $i_n^{X^\bullet} : \iota_n X^\bullet \to X^\bullet$, given by setting $(i_n^{X^\bullet})^m = \mathrm{Id}_{X^m}$ for $m \le n$, pictorially

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & X^n & \longrightarrow & X^{n-1} & \longrightarrow & \cdots . \\ & & \downarrow & & \| & & \| & & \\ \cdots & \longrightarrow & X^{n+1} & \longrightarrow & X^n & \longrightarrow & X^{n-1} & \longrightarrow & \cdots \end{array}$$

Then we have the following obvious lemma.

Lemma 5.4 *Let $f^\bullet : X^\bullet \to Y^\bullet$ be a homomorphism of complexes of A-modules. Assume that $f \circ i_n^{X^\bullet}$ is homotopic to 0. Then $f^\bullet$ is homotopic to some $g^\bullet$ such that $g^i = 0$ for all $i \le n$.*

In fact, by assumption, there exists a family of morphisms $\{h^i : X^i \to Y^{i+1}\}_{i \le n}$, such that $f^i = d_Y^{i+1} h^i + h^{i-1} d_X^i$ for all $i \le n$. We set $g^i = f^i$ for all $i > n+1$, $g^{n+1} = f^{n+1} - h^n d_X^{n+1}$, and $g^i = 0$ for all $i \le n$. Then $f^\bullet$ is homotopic to $g^\bullet$.

Now we can give the proof of the above proposition.

Proof of Proposition 5.3 Since A has finite global dimension, any $X^\bullet \in D^b(A\text{-Mod})$ has a projective resolution that is contained in $K^b(\mathrm{Proj}A)$, where $\mathrm{Proj}A$ is the additive category of projective A-modules. In other words, $K^b(\mathrm{Proj}A)$ is dense in $D^b(A\text{-Mod})$. Moreover, for any $P^\bullet \in K^b(\mathrm{Proj}A)$,

$$\mathrm{Hom}_{D^b(A\text{-Mod})}(P^\bullet, X^\bullet) = \mathrm{Hom}_{K^b(A\text{-Mod})}(P^\bullet, X^\bullet).$$

Thus given any $\zeta \in \mathrm{Z}^m(D^b(A\text{-Mod}))$, to show $\zeta = 0$ it suffices to show that $\zeta_{P^\bullet} = 0$ for any $P^\bullet \in K^b(\mathrm{Proj}A)$.

We use induction on the width of $P^\bullet$. Let $s(P^\bullet)$ and $l(P^\bullet)$ be the smallest and largest integer n such that $P^n \neq 0$ respectively. We say that $P^\bullet$ is supported in $[s(P^\bullet), l(P^\bullet)]$; and $w(P^\bullet) = l(P^\bullet) - s(P^\bullet) + 1$ is called the width of $P^\bullet$.

Since ζ is graded commutative with the shift functor [1], we need only to show that $\zeta_{P^\bullet} = 0$ for all $n \ge 0$ and those complexes in $K^b(\mathrm{Proj}A)$ which are supported in $[0, n]$, that is complexes of the form

$$\cdots \to 0 \to P^n \to P^{n-1} \to \cdots \to P^0 \to 0 \to \cdots .$$

Clearly $w(P^\bullet) = n+1$ in this case.

We deal with the cases $m < 0$ and $m > d$ separately.

Case 1: $m < 0$.

Obviously $\zeta_{P^\bullet} = 0$ if $w(n)+m \le 1$. We assume that $\zeta_{P^\bullet} = 0$ for all complexes $P^\bullet$ with $w(P^\bullet) \le n$. Now consider the following complex.

$$\cdots \to 0 \to P^n \to P^{n-1} \to \cdots \to P^0 \to 0 \to \cdots .$$

Since $w(\iota^{n-1} P^\bullet) \le n$, $\zeta_{\iota^{n-1} P^\bullet} = 0$ by assumption. Since ζ is a natural transformation, the composition map $\zeta_{P^\bullet} \circ i_{n-1}^{P^\bullet} = 0$ in $D^b(A)$, thus $\zeta_{P^\bullet} \circ i_{n-1}^{P^\bullet}$ is homotopic to zero. By Lemma 5.4, $\zeta_{P^\bullet}$ is homotopic to some $g^\bullet$ with $g^i = 0$ for all $i \le n-1$. Now g^n is automatically 0, and it follows that $g^\bullet = 0$, i.e., $\zeta_{P^\bullet}$ is 0 in $D^b(A\text{-Mod})$.

Case 2: $m > d$.

Clearly $\zeta_{P^\bullet} = 0$ for all complexes $P^\bullet \in K^b(\text{Proj}A)$ with $w(P^\bullet) \leq m$. We use induction on $w(P^\bullet)$. Assume that $\zeta_{P^\bullet} = 0$ for all complexes $P^\bullet$ with $w(P^\bullet) \leq m + n$, where $n \geq 0$. Now we consider a complex

$$P^\bullet\colon\ \cdots \to 0 \to P^{m+n} \to P^{m+n-1} \to \cdots \to P^0 \to 0 \to \cdots$$

with $w(P^\bullet) = m + n + 1$. We will show that $\zeta_{P^\bullet} = 0$.

Write $\zeta_{P^\bullet} = g^\bullet$ for short. Now consider the following map π :

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & P^{n+1} & \longrightarrow & P^n & \longrightarrow & P^{n-1} & \longrightarrow \cdots \longrightarrow & P^0 & \longrightarrow & \cdots, \\ & & & & \downarrow{\scriptstyle p^n} & & \| & & \| & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & \frac{P^n}{\text{Im}d^{n+1}} & \longrightarrow & P^{n-1} & \longrightarrow \cdots \longrightarrow & P^0 & \longrightarrow & 0 \end{array}$$

where p^n is the natural quotient map.

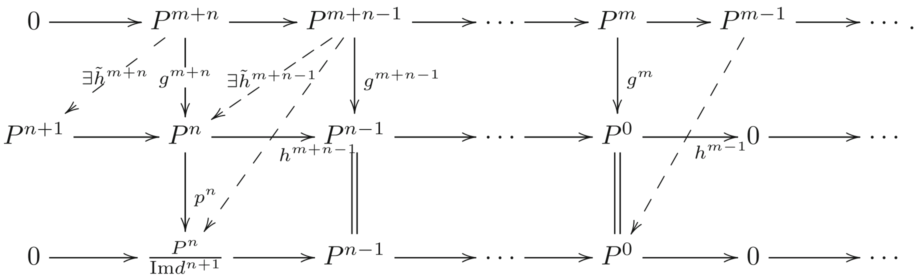
Since A has global dimension d, $\frac{P^n}{\text{Im}d^{n+1}}$ has a projective resolution of the form

$$\cdots \to 0 \to Q^{n+d} \to \cdots \to Q^{n+1} \to Q^n \to \frac{P^n}{\text{Im}d^{n+1}} \to 0.$$

Then we have a quasi-isomorphism

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & Q^{n+d} & \longrightarrow \cdots \longrightarrow & Q^n & \longrightarrow & P^{n-1} & \longrightarrow \cdots \longrightarrow & P^0 & \longrightarrow & 0\ . \\ & & & & \downarrow{\scriptstyle q^n} & & \| & & \| & & \\ & & \cdots & \longrightarrow 0 \longrightarrow & \frac{P^n}{\text{Im}d^{n+1}} & \longrightarrow & P^{n-1} & \longrightarrow \cdots \longrightarrow & P^0 & \longrightarrow & 0 \end{array}$$

Notice that the top complex has width strictly less than $n + m + 1$. By induction hypothesis, $\pi[m] \circ g^\bullet$ is zero in $D^b(A\text{-Mod})$ and hence homotopic to zero. Thus there exists a homotopy $h^\bullet\colon \pi[m] \circ g^\bullet \to 0$, pictorially



Let $\alpha^\bullet, \beta^\bullet\colon X^\bullet \to Y^\bullet$ be morphism of complexes. A homotopy $h^\bullet\colon \alpha^\bullet \to \beta^\bullet$ is by definition a family of morphisms $\{h^i : X^i \to Y^{i+1}\}_{i\in\mathbb{Z}}$ such that

$$d_{Y^\bullet}^{i+1} h^i + h^{i-1} d_{X^\bullet}^i = \alpha^i - \beta^i$$

for any $i \in \mathbb{Z}$.

Since p^n is surjective, h^{m+n-1} lifts to some $\tilde{h}^{m+n-1}\colon P^{m+n-1} \to P^{m+1}$. It is direct to show that $p^n(g^{m+n} - \tilde{h}^{m+n-1} d_{P^\bullet}^{m+n}) = 0$. Hence there exists some $\tilde{h}^{m+n}\colon P^{m+n} \to P^{n+1}$, such that $g^{m+n} = d^{n+1}\tilde{h}^{m+n} + \tilde{h}^{m+n-1} d_{P^\bullet}^{m+n}$.

Set $\tilde{h}^i = h^i$ for $i \neq m+n-1, m+n$. Then the family $\{\tilde{h}^i\}_{i\in\mathbb{Z}}$ gives a homotopy from $g^\bullet$ to 0, which means that $\zeta_{P^\bullet} = 0$ in $D^b(A\text{-Mod})$. The proof is competed. □

5.2 The Characteristic Morphism of a Semi-Simple Algebra

Example 5.5 We first recall an example from [6]. Let $\Bbbk$ be a field of characteristic p. Assume that $\Bbbk$ is not perfect. Then there exists some $a \in \Bbbk$, which has no p-th roots in $\Bbbk$. Consider the field extension $A = \Bbbk[a^{1/p}]$. Then $\mathrm{HH}^*(A)$ is isomorphic to $\Bbbk[x]$ as an algebra. Clearly the graded center of $D^b(A)$ is isomorphic to $\Bbbk$, thus the characteristic morphism is not injective in this case. In fact, we have a more general result.

Theorem 5.6 *Let A be a finite dimensional semi-simple algebra over $\Bbbk$. Then the following are equivalent:*

(1) *A is a projective A-bimodule;*
(2) *A is separable over $\Bbbk$;*
(3) *A is homologically smooth;*
(4) *the characteristic morphism χ of A is injective.*

Proof Note that the equivalence (1)$\Longleftrightarrow$(2) is well known for separable algebras and (1)$\Longrightarrow$(3) is obvious by definition.

By assumption A is semi-simple and the graded center of the derived category is concentrated in degree 0, then the characteristic morphism is injective if and only if $\mathrm{HH}^{\geq 1}(A) = 0$. Thus (1)$\Longrightarrow$(4) is clear.

Next we show that (4)$\Longrightarrow$(2). Assume that A is nonseparable. Combining Theorem 1.8.1 and Theorem 1.8.3 in [20], one shows that $\mathrm{HH}^1(A) \neq 0$. Since the graded center $\mathrm{Z}^*(D(A))$ is concentrated in degree 0, we obtain that $\mathrm{Ker}(\chi) \supseteq \mathrm{HH}^1(A) \neq 0$. Therefore, if the characteristic morphism of A is injective, then A must be separable.

We are only left to show the implication (3)$\Longrightarrow$(2). Notice that the separability condition and the homologically smooth condition are both preserved under Morita equivalence, which says that it suffices to prove (3)$\Longrightarrow$(2) for a finite dimensional division algebra D.

We need the following fact on separable algebras. Let K be the center of D. Then K is a finite field extension of $\Bbbk$ such that D is separable over K, and D is separable over $\Bbbk$ if and only if K is a separable extension, see for instance [12, Theorem 6.1.2].

Clearly D is a K-linear space, and hence a free K-module. Moreover, D is a direct sum of copies of K as $K \otimes_{\Bbbk} K$-modules, and $D \otimes_{\Bbbk} D$ is a free $K \otimes_{\Bbbk} K$-module. Now assume that D is homologically smooth over $\Bbbk$. Then D has a bounded resolution of projective $D \otimes_{\Bbbk} D$-modules, which is also a projective resolution of D as $K \otimes_{\Bbbk} K$-modules. Thus the projective dimension of D as a K-bimodule is finite, and so is K. In particular, $\mathrm{HH}^n(K, K)$ vanish for sufficiently large n, therefore K is a separable extension of $\Bbbk$ by [3, Corollary], see also [16]. It follows that D is separable over $\Bbbk$, and the proof is completed. □

5.3 The Characteristic Morphism of a Path Algebra

Now we turn to path algebras, a special class of hereditary algebras. Note that a semi-simple algebra is an algebra of global dimension 0, and hereditary algebras, those algebras of global dimension 1, are the simplest algebras next to semi-simple ones in some sense. We mention that a path algebra is always homologically smooth. The main result is stated as follows.

Theorem 5.7 *Let $A = \Bbbk Q$ be the path algebra of a finite acyclic quiver. Then the characteristic morphism of A is injective.*

Proof By [23, Lemma 3.1] and Remark 4.1(3), it suffices to show that χ is injective when restricted to $\mathrm{HH}^1(A)$. That is, for any outer derivation $\delta \in \mathrm{Der}(A) \setminus \mathrm{InDer}(A)$, we need to show that $\chi(\bar{\delta}) \neq 0$. There are two cases.

Case 1. $\delta(a) \notin \langle a \rangle$ for some $a \in A$.

This case is easy. In fact, by Lemma 3.6, there exists some A-module M such that the sequence $0 \to M \xrightarrow{\binom{1}{0}} M \oplus_{\delta} M \xrightarrow{(0,1)} M \to 0$ is non-split. Then by Proposition 3.4, we have $\chi(\bar{\delta}) \neq 0$.

Case 2. $\delta(a) \in \langle a \rangle$ for all $a \in A$.

Let Γ be a spanning forest of Q. By Proposition 4.6, there exists some δ', which is a linear combination of elements in X_Γ with $\bar{\delta} = \bar{\delta}'$, where X_Γ denotes the set $\{\bar{\partial}_\alpha^p \mid \alpha \in Q_1, p \in Q, p \parallel \alpha\} \setminus \{\bar{\partial}_\alpha^\alpha | \alpha \in \Gamma_1\}$. Clearly $\delta'(a) \in \langle a \rangle$ for all $a \in A$.

For any $\alpha \in Q_1$, set V_α to be the subspace spanned by $\{p \in Q \mid p \parallel \alpha\}$. Since Q is acyclic, obviously $\langle \alpha \rangle \cap V_\alpha = \Bbbk\alpha$, thus $\delta'(\alpha) = \lambda_\alpha \alpha$ for some $\lambda_\alpha \in \Bbbk$. By construction of X_Γ, we know that $\delta'(\alpha) = 0$ for any $\alpha \in \Gamma_1$.

Obviously $\delta' \neq 0$ by the assumption that δ is outer. Then there exists some arrow $\beta \in Q_1$ such that $\delta'(\beta) \neq 0$, that is, $\delta'(\beta) = \lambda\beta$ for some $\lambda \neq 0$. Let I be the left ideal of A generated by the set

$$\{\alpha - e_{t(\alpha)} \mid \alpha \in \Gamma_1, \text{ or } \alpha = \beta\} \cup \{\gamma \mid \gamma \notin \Gamma_1, \gamma \neq \beta\}.$$

Consider the left A-module $M = A/I$. It is direct to show that $M = \bigoplus_{v \in Q_0} \Bbbk\bar{e}_v$.

We claim that there exists no $\Bbbk$-linear map $f\colon M \to M$, such that for any $a \in A$ and $m \in M$, $f(am) = af(m) + \delta'(a)m$. Otherwise, let f be such a map. Then for any $v \in Q_0$,

$$f(\bar{e}_v) = f(e_v\bar{e}_v) = e_v f(\bar{e}_v) + \delta'(e_v)\bar{e}_v = e_v f(\bar{e}_v) = \lambda_v \bar{e}_v$$

for some $\lambda_v \in \Bbbk$. Moreover, for any $\alpha \in \Gamma_1$, we have

$$f(\bar{e}_{t(\alpha)}) = f(\bar{\alpha}) = f(\alpha\bar{e}_{s(\alpha)}) = \alpha f(\bar{e}_{s(\alpha)}) + \delta'(\alpha)\bar{e}_{s(\alpha)} = \alpha f(\bar{e}_{s(\alpha)}),$$

it follows that $\lambda_{s(\alpha)} = \lambda_{t(\alpha)}$.

By the maximality of a spanning forest, there exist arrows $\alpha_1, \cdots, \alpha_r \in \Gamma_1$, such that the set $\{\beta, \alpha_1, \cdots, \alpha_r\}$ forms a cycle. Thus we have $\lambda_{s(\beta)} = \lambda_{t(\beta)}$. On the other hand side,

$$\lambda_{t(\beta)}\bar{e}_{t(\beta)} = f(\bar{e}_{t(\beta)}) = f(\bar{\beta}) = f(\beta\bar{e}_{s(\alpha)}) = \beta f(\bar{e}_{s(\beta)}) + \delta'(\beta)\bar{e}_{s(\beta)}) = (\lambda_{s(\beta)} + \lambda)\bar{e}_{t(\beta)},$$

it forces that $\lambda = 0$, which leads to a contradiction.

Thus by Lemma 3.5 the sequence $0 \to M \xrightarrow{\binom{1}{0}} M \oplus_{\delta'} M \xrightarrow{(0,1)} M \to 0$ is non-split, again by Proposition 3.4, we have $\chi(\bar{\delta}) \neq 0$.

Now all possible cases have been exhausted and we are done. □

We mention that our argument does not apply to an arbitrary hereditary algebra, the reason is that in general case, we do not have a nice description of elements in the Hochschild cohomology groups. However, if the field $\Bbbk$ is algebraically closed, then we get more. Note that the Hochschild cohomology and the derived category are both invariant under derived equivalence, and especially invariant under Morita equivalence. Combined with the fact that over an algebraically closed field, any finite dimensional hereditary algebra is Morita equivalent to a path algebra of a finite acyclic quiver, we draw the following consequence.

Corollary 5.8 *Let* $\Bbbk$ *be an algebraically closed field and* A *a finite dimensional hereditary algebra. Then the characteristic morphism of* A *is injective.*

5.4 A Further Example: the Polynomial Ring $\Bbbk[x]$

Example 5.9 We end with an example of a finite quiver with oriented cycles. Consider the quiver with exactly one vertex and one loop attached. The path algebra of this quiver is $\Bbbk[x]$, the polynomial ring in one variable. We claim that in this case, the characteristic morphism is also injective.

Clearly any nonzero derivation of $\Bbbk[x]$ is outer. It is well known that any derivation δ of $\Bbbk[x]$ is uniquely determined by its value $\delta(x)$ at the monomial x, and δ is nonzero if and only if $\delta(x)$ is nonzero. In fact, $\delta(f(x)) = f'(x)\delta(x)$ for any $f(x) \in \Bbbk[x]$, where $f'(x)$ is the formal derivative of $f(x)$, which is defined by

$$(a_n x^n + \cdots + \cdots a_1 x + a_0)' = n a_n x^{n-1} + \cdots + 2a_2 x + a_1.$$

Let δ be a nonzero derivation of $\Bbbk[x]$. We first show that there exists some irreducible polynomial $h(x) \in \Bbbk[x]$ with $h'(x) \neq 0$ and $\delta(x) \notin h(x)\Bbbk[x]$. If $\Bbbk$ is an infinite field, then $h(x)$ can be chosen to be of the form $x + \lambda$ for some λ in $\Bbbk$. Now assume that $\Bbbk$ is a finite field. Let n be a positive integer which is strictly greater than the degree of $\delta(x)$ and coprime to the characteristic of $\Bbbk$. A well-known result on finite fields says that there exists at least one irreducible polynomial $h(x) \in \Bbbk[x]$ of degree n. Clearly $h'(x)$ has degree $n - 1$ and hence $h'(x) \neq 0$. Moreover, we have $\delta(x) \notin h(x)\Bbbk[x]$ for the degree of $\delta(x)$ is strictly less that the one of $h(x)$.

Similarly by comparing the degrees of $h(x)$ and $h'(x)$ we know that $h'(x) \notin h(x)\Bbbk[x]$. Since $h(x)$ is irreducible, $h(x)\Bbbk[x]$ is a maximal ideal and hence a prime ideal. It follows that $\delta(h(x)) = h'(x)\delta(x) \notin h(x)\Bbbk[x]$, and by the same argument as in the proof of Theorem 5.7, we obtain that $\chi(\bar{\delta}) \neq 0$.

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References

1. Auslander, M., Reiten, I., Smalø, S.: Representation theory of Artin algebras, Cambridge Studies in Adv. Math. 36 Cambridge Univ Press (1995)
2. Avramov, L.L., Buchweitz, R.-O.: Support varieties and cohomology over complete intersections. Invent. Math. **142**, 285C318 (2000)
3. Avramov, L.L., Iyengar, S.: Gaps in Hochschild cohomology imply smoothness for commutative algebras. Math. Res. Lett. **12**, 789–804 (2005)
4. Avramov, L.L., Iyengar, S.: Modules with prescribed cohomological support. Ill. J. Math. **51**, 1–20 (2007)
5. Benson, D., Iyengar, S., Krause, H.: Local cohomology and support for triangulated categories. Ann. Sci. Éc. Norm. Supér. (4) **41**(4), 573–619 (2008)
6. Bergh, P.A., Iyengar, S., Krause, H., Oppermann, S.: Dimensions of triangulated categories via Koszul objects. Math. Z **265**(4), 849–864 (2010)
7. Buchweitz, R.O., Flenner, H.: Hochschild (co-)homology of singular spaces. Adv. Math. **217**(1), 205–242 (2008)

8. Buchweitz, R.O., Flenner, H.: The global decomposition theorem for Hochschild (co-)homology of singular spaces via the Atiyah-Chern character. Adv. Math. **217**(1), 243C281 (2008)
9. Buchweitz, R.O., Green, E.L., Madsen, D., Solberg, Ø.: Finite Hochschild cohomology without finite global dimension. Math. Res. Lett. **12**, 805–16 (2005)
10. Cartan, H., Eilenberg, S.: Homological algebra, Princeton Mathematical Series 19 Princeton University Press (1956)
11. Cibils, C.: On the Hochschild cohomology of finite dimensional algebras. Comm. Algebra **16**, 645–649 (1988)
12. Drozd, Y.A., Kirichenko, V.V.: Finite Dimensional Algebras. Springer-Verlag, New York (1994)
13. Gerstenhaber, M.: The cohomology structure of an associative ring. Ann. Math. **78**(2), 267–288 (1963)
14. Gerstenhaber, M.: On the deformation of rings and algebras. Ann. Math. **79**(2), 59–103 (1964)
15. Guo, L., Li, F.: Structure of Hochschild cohomology of path algebras and differential formulation of Euler's polyhedron formula. Asian J. Math. **18**(3), 545–572 (2014)
16. Han, Y.: Hochschild (co)homology dimension. J. Lond. Math. Soc. **73**(2), 657–668 (2006)
17. Happel, D.: Triangulated categories in the representation theory of finite-dimensional algebras, LMS Lecture Note Series 119, Cambridge University Press, Cambridge, 1988. x+208 pp. ISBN: 0-521-33922-7
18. Happel, D.: Hochschild cohomology of finite-dimensional algebras Lect. Notes Math., vol. 1404, pp. 108–126. Springer-Verlag, Berlin (1989)
19. Hochschild, G.: On the cohomology groups of an associative algebra. Ann. Math. **46**, 58–67 (1945)
20. Jacobson, N.: Finite dimensional division algebras over fields Springer-Verlag (1996)
21. Keller, B.: Derived invariance of higher structures on the Hochschild complex Preprint. http://www.math.jussieu.fr/keller/publ/dih.dvi (2003)
22. Krause, H.: Derived Categories, Resolutions, and Brown Representability. In: Interactions between homotopy theory and algebra, 101C139, Contemp. Math. 436, Amer. Math. Soc., Providence, RI, (2007)
23. Krause, H., Ye, Y.: On the centre of a triangulated category. Proc. Edinb. Math. Soc **54**(2), 443–466 (2011)
24. Künzer, M.: On the center of the derived category preprint (2006)
25. Linckelmann, M.: On graded centres and block cohomology. Proc. Edinb. Math. Soc **52**(2), 489–514 (2009)
26. Linckelmann, M., Stancu, R.: On the graded center of the stable category of a finite pgroup. J. Pure Appl. Algebra **214**(6), 950–959 (2010)
27. Lowen, W.: Hochschild cohomology, The characteristic morphism and derived deformations. Compos. Math. **144**(6), 1557–1580 (2008)
28. Lowen, W., Van den Bergh, M.: Hochschild cohomology of abelian categories and ringed spaces. Adv. Math. **198**(1), 172–221 (2005)
29. Rickard, J.: Derived equivalences as derived functors. J. Lond. Math. Soc. **43**(2), 37–48 (1991)
30. Ringel, C.M.: Tame algebras and integral quadratic forms Lect. Notes Math., vol. 1099. Springer-Verlag, Berlin (1984)
31. Snashall, N., Solberg, Ø.: Support varieties and Hochschild cohomology rings. Proc. Lond. Math. Soc. **88**, 705–32 (2004)
32. Verdier, J.L.: Des catégories dérivées des catégories abéliennes. Astérisque, 239 (1996)
33. Weibel, C.A.: An Introduction to Homological Algebra Cambridge Studies in Advanced Mathematics, vol. 38. Cambridge University Press, Cambridge (1994)
34. Zhang, P.: Hochschild cohomology of a truncated basic cycle. Sci. in China (A) **40**(12), 1272–1278 (1997)